# Vector Fields

**Def$^n$:** A <u>vector field</u> on $\mathbb{R}^n$ is a function $\vec{v} : \mathbb{R}^n \to \mathbb{R}^n$.

We often visualize vector fields as attatching vectors to points.

NB: GeoGebra is a useful tool for visualizing vector fields.

**Ex:** Illustrate the vector fields below in GeoGebra:

1. $\vec{v}(x,y) = \langle x, 0 \rangle$ ←
2. $\vec{v}(x,y) = \langle 0, x \rangle$ ←
3. $\vec{v}(x,y) = \langle y, x \rangle$ ←
4. $\vec{v}(x,y) = \langle x, y \rangle$ ←
5. $\vec{v}(x,y) = \langle x^2 + xy, xy - 3 \rangle$ ←

**Ex:** Given a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, the gradient $\nabla f$ is a vector field on $\mathbb{R}^n$.

↳ For $f(x,y,z) = x^3 y + 3xz - yz$, the * <u>gradient vector field</u> of $f$ is $\vec{\nabla} f(x,y,z) = \langle 3x^2 y + 3z, x^3 - z, 3x - z \rangle$

**Def$^n$:** A vector field $\vec{v} : \mathbb{R}^n \to \mathbb{R}^n$ is <u>conservative</u> when there exists a <u>potential function</u> $f : \mathbb{R}^n \to \mathbb{R}$ satisfying $\vec{v} = \vec{\nabla} f$.

**Q:** Which vector fields are conservative?

**Prop (Test for Conservativity):** Suppose vector field $\vec{V}(x_1, x_2, \cdots, x_n)$ has component functions $V_1, V_2, \cdots, V_n$ so that the partial derivatives of $V_1, \cdots, V_n$ are all continuous. The vector field $\vec{V}$ is conservative if and only if we have

$$\frac{\partial V_i}{\partial x_j} = \frac{\partial V_j}{\partial x_i} \quad \text{for all} \quad 1 \le i < j \le n.$$

IDEA $\longrightarrow$ $f_{x_i x_j} = f_{x_j x_i}$ $\leftarrow$ Clairaut's Theorem

This theorem can be roughly restated as follows:

a vector field is conservative precisely when it satisfies the conclusion of Clairaut's Theorem.

When $\vec{v}(x,y) = \langle P(x,y), Q(x,y) \rangle$, the theorem reduces to the claim $P_y = Q_x$ (provided these are continuous...).

**Ex:** Is $\vec{v}(x,y) = \langle \underset{P}{2xy}, \underset{Q}{x^2 - 3y^2} \rangle$ conservative?

**Sol:** $P_y = \frac{\partial}{\partial y}[2xy] = 2x$ and $Q_x = \frac{\partial}{\partial x}[x^2 - 3y^2] = 2x$

$\therefore P_y = Q_x$ so $\vec{v}$ is conservative. //

**Ex:** Is $\vec{v}(x,y) = \langle x+y, x-y \rangle$ conservative?

**Sol:** $P_y = \frac{\partial}{\partial y}[x+y] = 1$ and $Q_x = \frac{\partial}{\partial x}[x-y] = 1$

$\therefore P_y = Q_x$ yields $\vec{v}$ is conservative. //

Ex: Which of the following are conservative?

① $\vec{v}(x,y) = \langle xy, xy \rangle$ Is $P_y = Q_x$ ?

Sol: $P_y = \frac{\partial}{\partial y}[xy] = x$ and $Q_x = \frac{\partial}{\partial x}[xy] = y$

$\therefore P_y \neq Q_x$ yields $\vec{v}$ is not conservative. ☒

② $\vec{v}(x,y) = \langle 3x^3 - 4y, 4x - y \rangle$

Sol: $\frac{\partial}{\partial y}[3x^3 - 4y] = -4$ $\frac{\partial}{\partial x}[4x - y] = 4$

$\therefore P_y \neq Q_x$ yields $\vec{v}$ is not conservative. ☒

③ $\vec{v}(x,y) = \langle 6xy - 4y, 3x^2 - 4x + 7 \rangle$

Sol: $P_y = \frac{\partial}{\partial y}[6xy - 4y] = 6x - 4$ and $Q_x = \frac{\partial}{\partial x}[3x^2 - 4x + 7] = 6x - 4$

$\therefore P_y = Q_x$ yields $\vec{v}$ is conservative. ☒

④ $\vec{v}(x,y) = \langle y\cos(xy), x\cos(xy) \rangle$

Sol: $P_y = \frac{\partial}{\partial y}[y\cos(xy)] = y \cdot -x\sin(xy) + 1 \cdot \cos(xy)$
$= \cos(xy) - xy\sin(xy)$

$Q_x = \frac{\partial}{\partial x}[x\cos(xy)] = x \cdot -y\sin(xy) + 1 \cdot \cos(xy)$
$= \cos(xy) - xy\sin(xy)$

$\therefore P_y = Q_x$ yields $\vec{v}$ is conservative. ☒

**NB**: While the proof of the conservativity test is not difficult, it does require Mathematical Induction, so we omit it...

**Q**: Given a conservative vector field $\vec{v}$, how can we compute a potential function for $\vec{v}$? ☆

**A**: Take advantage of Clairaut's Theorem!

↳ The solutions below imitate the proof of the conservativity test...

**Ex**: Compute a potential for $\vec{v}(x,y) = \langle 2xy^3 - 3, 3x^2y^2 + 5 \rangle$ if it exists.

**Sol**: $\frac{\partial}{\partial y}[2xy^3 - 3] = 6xy^2$ and $\frac{\partial}{\partial x}[3x^2y^2 + 5] = 6xy^2$

$\therefore \vec{v}$ is conservative. $\therefore \vec{v} = \nabla f$ for some $f(x,y)$.

$$f(x,y) = \int \frac{\partial f}{\partial x}\,dx = \int (2xy^3 - 3)\,dx = x^2y^3 - 3x + C(y)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[x^2y^3 - 3x + C(y)] = 3x^2y^2 + C'(y)$$

$$Q(x,y) = 3x^2y^2 + 5 \quad \therefore C'(y) = 5$$

$$C(y) = \int C'(y)\,dy = \int 5\,dy = 5y + D$$

$\therefore$ We obtain potential function $f(x,y) = x^2y^3 - 3x + 5y + D$ //

**Ex**: Compute a potential for $\vec{v}(x,y) = \langle \ln(xy), e^{xy} \rangle$ if one exists.

**Sol**: $P_y = \frac{\partial}{\partial y}[\ln(xy)] = \frac{x}{xy} = \frac{1}{y}$ and $Q_x = \frac{\partial}{\partial x}[e^{xy}] = ye^{xy}$

$P_y \neq Q_x$ so there is no potential function. ///

**Ex:** Compute a potential of $\vec{v} = \langle x, y \rangle$ if one exists.

**Sol:** $\frac{\partial}{\partial y}[x] = 0$ and $\frac{\partial}{\partial x}[y] = 0$

$\therefore \vec{v} = \nabla f$ for some potential $f(x,y)$.

$$f(x,y) = \int \frac{\partial f}{\partial x}\, dx = \int x\, dx = \frac{1}{2}x^2 + C(y)$$

$$\therefore \quad y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\tfrac{1}{2}x^2 + C(y)\right] = c'(y).$$

Hence $C(y) = \int c'(y)\, dy = \int y\, dy = \frac{1}{2}y^2 + D$

$$\therefore \quad f(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + D$$

$\blacksquare$

**Ex:** Compute a potential for $\vec{v}(x,y) = \left\langle \ln(y) + \frac{y}{x}, \ln(x) + \frac{x}{y} \right\rangle$ provided it exists.

**Sol:** $\frac{\partial}{\partial y}\left[\ln(y) + \frac{y}{x}\right] = \frac{1}{y} + \frac{1}{x}$ and $\frac{\partial}{\partial x}\left[\ln(x) + \frac{x}{y}\right] = \frac{1}{x} + \frac{1}{y}$

$\therefore P_y = Q_x$ yields $\underline{\vec{v} = \nabla f}$ for some $f(x,y)$.

$$f(x,y) = \int \frac{\partial f}{\partial x}\, dx = \int \left(\ln(y) + \frac{y}{x}\right) dx = x\ln(y) + y\ln(x) + C(y)$$

$$\therefore \quad \ln(x) + \frac{x}{y} = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[x\ln(y) + y\ln(x) + C(y)\right] = \frac{x}{y} + \ln(x) + c'(y)$$

$\therefore c'(y) = 0 \qquad \therefore C(y) = \int c'(y)\, dy = \int 0\, dy = D$

Hence $f(x,y) = x\ln(y) + y\ln(x) + D$ for some constant $D$ $\blacksquare$

**Ex:** Compute a potential function (if it exists) of vector field

$$\vec{v}(x,y,z) = \langle 2x+3y-yz,\ 3x-xz+z,\ -xy+y-1 \rangle$$

**Sol:** Check Conservativity: Write $\vec{v} = \langle v_x, v_y, v_z \rangle$.

$$\frac{\partial v_x}{\partial y} = \frac{\partial}{\partial y}\left[2x+3y-yz\right] = 3-z = \frac{\partial}{\partial x}\left[3x-xz+z\right] = \frac{\partial v_y}{\partial x}$$

$$\frac{\partial v_x}{\partial z} = \frac{\partial}{\partial z}\left[2x+3y-yz\right] = -y = \frac{\partial}{\partial x}\left[-xy+y-1\right] = \frac{\partial v_z}{\partial x}$$

$$\frac{\partial v_y}{\partial z} = \frac{\partial}{\partial z}\left[3x-xz+z\right] = 1-x = \frac{\partial}{\partial y}\left[-xy+y-1\right] = \frac{\partial v_z}{\partial y}$$

$\therefore\ \vec{v} = \nabla f$ for some $f(x,y,z)$ (i.e. $\vec{v}$ is conservative ☺).

$$f(x,y,z) = \int \frac{\partial f}{\partial x}\,dx = \int (2x+3y-yz)\,dx = x^2+3xy-xyz+C(y,z)$$

$$3x-xz+z = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[x^2+3xy-xyz+C(y,z)\right] = 3x-xz+C_y(y,z)$$

$\therefore\ C_y(y,z) = z$, so $C(y,z) = \int C_y(y,z)\,dy = \int z\,dy = yz + D(z)$.

$\therefore\ f(x,y,z) = x^2+3xy-xyz+yz+D(z)$

$$-xy+y-1 = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[x^2+3xy-xyz+yz+D(z)\right] = -xy+y+D'(z)$$

$\therefore\ D'(z) = -1$ so $D(z) = \int D'(z)\,dz = \int -1\,dz = -z+E$

Hence $f(x,y,z) = x^2+3xy-xyz+yz-z+E$ is a potential. ▨

Ex: Compute a potential function of vector field $\vec{v}(x,y,z) = \langle 2x+y,\ x+3,\ 3y-1 \rangle$ provided it exists.

Sol: Check conservativity. Let $\vec{v} = \langle v_x, v_y, v_z \rangle$.

$$\frac{\partial v_x}{\partial y} = \frac{\partial}{\partial y}[2x+y] = 1 = \frac{\partial}{\partial x}[x+3] = \frac{\partial v_y}{\partial x}$$

$$\frac{\partial v_x}{\partial z} = \frac{\partial}{\partial z}[2x+y] = 0 = \frac{\partial}{\partial x}[3y-1] = \frac{\partial v_z}{\partial x}$$

$$\frac{\partial v_y}{\partial z} = \frac{\partial}{\partial z}[x+3] = 0 \neq 3 = \frac{\partial}{\partial y}[3y-1] = \frac{\partial v_z}{\partial y}$$

$\therefore \vec{v}$ is not conservative.

Hence $\vec{v}$ does not admit a potential function $\square$